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THE EFFECT OF COUPLE-STRESSES ON SINGULARITIES DUE TO DISCONTINUOUS LOADINGS*

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Abstract—The singularities induced by discontinuous normal or shearing tractions applied to a semi-infinite solid are examined within the two-dimensional linearized couple-stress theory of elastic behavior and are compared with their counterpart in the classical theory of plane strain. A finite jump in the *shear* loading is found to produce a logarithmic infinity at the load discontinuity in the *normal* stress acting parallel to the boundary according to both theories. In contrast, a finite jump in the *normal* loading, according to the couple-stress theory, gives rise to a logarithmic infinity at the load discontinuity in the *shear* stress at right angles to the boundary, although all stresses remain bounded in the corresponding classical solution; whereas the conventional theory for this loading case predicts a logarithmic singularity in the rotation field, the latter remains bounded in the modified solution.

INTRODUCTION

THIS study is a sequel to two previous investigations [1, 2] concerning the influence of couple-stresses upon singular concentrations of stress in elastic solids. The linearized couple-stress theory of elastic behavior underlying [1, 2] was explored comprehensively by Mindlin and Tiersten[3], while Mindlin [4] considered separately the corresponding two-dimensional theory of plane strain.

The specific singular plane-strain problems treated in [1] are those of the half-plane subjected to a concentrated normal or tangential edge load, the half-plane under *shearing* tractions uniformly distributed over a finite segment of the boundary, and the problem of the half-plane indented by a smooth flat punch. On the other hand, [2] deals with the concentration of stress around a finite straight crack in a transverse field of uni-axial tension.

Both [1] and [2] were motivated by results in [3, 4] that display a mitigating effect of couple-stresses upon the stress concentration due to a circular hole in a uniform, non-isotropic field of stress. This finding suggested a question regarding singular problems for which the conventional theory predicts unbounded concentrations of stress, accompanied by locally infinite deformations: to what extent are such pathological predictions altered—or possibly even eliminated—by the couple-stress theory, which assigns an explicit role to the rotation gradients in its governing constitutive law?

The conclusions reached in connection with the foregoing question on the basis of the particular singular problems studied in [1, 2] may be summarized as follows:

(a) The *rotation field* in each instance remains bounded according to the couple-stress theory even if the corresponding classical solution exhibits a locally infinite rotation;

(b) The *couple-stress field* in all cases considered either remains bounded or displays singularities of an order not exceeding that which is characteristic of the ordinary stress field appropriate to the modified solution;

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(c) The *ordinary stress field* in the modified solution possesses singularities of the same order—though not of the same detailed structure—as in the conventional solution, and individual ordinary stresses that are bounded in the classical solution remain bounded in the presence of couple-stresses.

It is the main purpose of this paper to show that conclusion (c) is not universally valid: a singular problem whose conventional treatment leads to finite stresses may, within the framework of the couple-stress theory, give rise to a locally unbounded concentration of stress. The example used to establish this claim is supplied by the half-plane under distributed *normal* tractions with a finite jump discontinuity. The solution deduced for this problem reveals a logarithmic infinity at the load discontinuity in the shear stress acting perpendicular to the boundary, although the entire stress field is bounded in the associated conventional solution. This result, which is independent of the size of the material length-parameter, would appear to be of particular relevance to an assessment of the as yet dubious physical status of the couple-stress theory.

In Section 1 we recall briefly only those features of the modified and the classical theory of plane strain that are pertinent to our specific purpose. In the same section we also cite from [1] the modified solution to the problem of the half-plane under anti-symmetric normal or shearing tractions. This solution is used in Section 2 to examine in detail the asymptotic character of the rotation, stress, and couple-stress fields in the vicinity of a finite discontinuity of the normal or shear loading. The results thus obtained are compared with their counterpart in the conventional theory. Although the present more general and more detailed treatment of discontinuous shearing tractions yields information that could in part have been anticipated from the solution for a *uniformly* distributed discontinuous shear loading given in [1], it has been included here for comparison purposes.

1. PLANE STRAIN IN THE COUPLE-STRESS AND THE CLASSICAL THEORY. THE HALF-PLANE UNDER DISTRIBUTED NORMAL OR SHEAR LOADING

In order to render this paper reasonably self-contained we recall here first—in a limited form adapted to our present needs—the formulation of the relevant class of boundary-value problems in both the linearized couple-stress and the classical equilibrium theory of plane strain for a homogeneous and isotropic elastic solid. In this connection we adhere to the notation used in [1, 2]. Thus, $\mathbf{u}$ and $\boldsymbol{\omega}$ denote the displacement and rotation vector fields; $\mathbf{e}$ and $\boldsymbol{\kappa}$ are the infinitesimal strain and the curvature-twist tensor fields; $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ designate the tensor fields of conventional stress and couple-stress, respectively. Further, we refer the governing equations to rectangular cartesian coordinates (x_1, x_2, x_3) and employ the usual indicial notation with the agreement that Latin and Greek subscripts have the respective ranges (1, 2, 3) and (1, 2).

Suppose now the solid under consideration occupies a cylindrical or prismatic region R of three-space and call D, with the boundary C, the open cross-section of R. If the coordinate frame is chosen so that the x_3-axis is parallel to the generators of R, the assumption of plane deformations parallel to the plane $x_3 = 0$ takes the form

$$u_{\alpha,3} = 0, \quad u_3 = 0 \text{ on } R. \tag{1.1}$$

Entering the fundamental field equations of the *modified theory* with (1.1) and adopting the normalization of the couple-stress field*

* Recall that the isotropic part of $\boldsymbol{\sigma}$ remains indeterminate.

$$\sigma_{kk} = 0 \text{ on } R, \tag{1.2}$$

one finds that all component fields are independent of x_3 and throughout D obey

$$\left.\begin{aligned} \omega_\alpha = e_{3i} = e_{i3} = \kappa_{3i} = \kappa_{\alpha\beta} = 0, \\ \tau_{3\alpha} = \tau_{\alpha 3} = 0, \quad \sigma_{33} = 0, \quad \sigma_{\alpha\beta} = 0 \quad (\alpha \neq \beta). \end{aligned}\right\} \tag{1.3}$$

Next introduce the abridged symbols

$$\omega = \omega_3, \qquad \kappa_\alpha = \kappa_{\alpha 3}, \qquad \sigma_\alpha = \sigma_{\alpha 3}, \tag{1.4}$$

call $\varepsilon_{\alpha\beta}$ the components of the two-dimensional alternator and let $\delta_{\alpha\beta}$ be the Kronecker-delta. The complete system of two-dimensional field equations appropriate to the modified plane-strain theory, consisting of the kinematic relations, the constitutive relations, and the stress equations of equilibrium, then becomes

$$\left.\begin{aligned} \omega = \tfrac{1}{2}\varepsilon_{\alpha\beta}u_{\beta,\alpha}, \qquad e_{\alpha\beta} = u_{(\alpha,\beta)}, \qquad \kappa_\alpha = \omega_{,\alpha}, \\ e_{\alpha\beta} = \frac{1}{2\mu}[\tau_{(\alpha\beta)} - \nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad 4\mu l^2\kappa_\alpha = \sigma_\alpha, \\ \tau_{\beta\alpha,\beta} = 0, \qquad \varepsilon_{\alpha\beta}\tau_{\alpha\beta} + \sigma_{\alpha,\alpha} = 0, \end{aligned}\right\} \tag{1.5}$$

where the constants μ, ν, and l—in this order—designate the shear modulus, Poisson's ratio, and the characteristic length-parameter of the material at hand. As is apparent, we assume the body-force and body-couple fields to vanish identically. Equations (1.5) hold on D and, in the case of given *loads* to which we confine our attention, are accompanied by the boundary conditions

$$\tau_{\beta\alpha}n_\beta = t_\alpha, \qquad \sigma_\alpha n_\alpha = s \text{ on } C. \tag{1.6}$$

Here t_α stands for the given components of the ordinary traction vector, s refers to the pre-assigned axial component of the couple-traction vector, whereas n_α denotes the components of the unit outward normal of C.

Elimination of all kinematic field quantities among (1.5) furnishes a characterization of $\tau_{\alpha\beta}$ and σ_α in terms of the stress equations of equilibrium and compatibility

$$\left.\begin{aligned} \tau_{\beta\alpha,\beta} = 0, \qquad \varepsilon_{\alpha\beta}\tau_{\alpha\beta} + \sigma_{\alpha,\alpha} = 0, \\ \varepsilon_{\alpha\beta}\sigma_{\alpha,\beta} = 0, \qquad \sigma_\alpha = 2l^2[\varepsilon_{\beta\gamma}\tau_{(\alpha\gamma),\beta} + \nu\varepsilon_{\alpha\beta}\tau_{\gamma\gamma,\beta}]. \end{aligned}\right\} \tag{1.7}$$

Further, (1.7) in conjunction with (1.6) suffice to determine $\tau_{\alpha\beta}$ and σ_α completely if D is simply connected. In this event the associated displacement and rotation fields are assured of being single-valued on D and are obtainable by integration of the partial differential equations

$$u_{(\alpha,\beta)} = \frac{1}{2\mu}[\tau_{(\alpha\beta)} - \nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad 4\mu l^2\omega_{,\alpha} = \sigma_\alpha, \tag{1.8}$$

which follow at once from (1.5). Finally, we recall that under suitable regularity assumptions the solution to the foregoing two-dimensional boundary-value problem is unique, as far as the stresses and couple-stresses are concerned, if the elastic constants μ, ν, and l satisfy the inequalities

$$\mu > 0, \qquad -1 < \nu < \tfrac{1}{2}, \qquad l > 0, \tag{1.9}$$

which guarantee the positive definiteness of the strain-energy density.

In the *classical theory* of plane strain the couple-stress field is assumed to vanish identically and (1.5), (1.6) give way to

$$e_{\alpha\beta} = u_{(\alpha,\beta)}, \qquad e_{\alpha\beta} = \frac{1}{2\mu}[\tau_{\alpha\beta} - \nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad \tau_{\beta\alpha,\beta} = 0, \tag{1.10}^*$$

$$\tau_{\beta\alpha}n_\beta = t_\alpha \qquad \text{on } C, \tag{1.11}$$

whereas (1.7), (1.8) are to be replaced by

$$\tau_{\beta\alpha,\beta} = 0, \qquad \tau_{\beta\alpha} = \tau_{\alpha\beta}, \qquad \tau_{\alpha\alpha,\beta\beta} = 0, \tag{1.12}$$

$$u_{(\alpha,\beta)} = \frac{1}{2\mu}[\tau_{\alpha\beta} - \nu\delta_{\alpha\beta}\tau_{\gamma\gamma}], \qquad 2\mu\omega_{,\alpha} = (1-\nu)\varepsilon_{\beta\alpha}\tau_{\gamma\gamma,\beta}. \tag{1.13}$$

As for the transition from the modified to the classical theory, we note from (1.5) that $\sigma_\alpha \to 0$ as $l \to 0$, provided κ_α remains bounded in this limit. In these circumstances (1.5), (1.6) yield (1.10), (1.11)—and hence also (1.12), (1.13)—as the characteristic length-parameter approaches zero.

Consider next the particular plane-strain problem of the half-plane under given surface tractions. To this end let D from now on stand for the open upper half-plane, and call $\bar{D}$ and C the closure and the boundary of D, respectively (Fig. 1). Thus,

$$\left.\begin{aligned} D &= \{(x_1, x_2) | -\infty < x_1 < \infty,\ 0 < x_2 < \infty\}, \\ \bar{D} &= \{(x_1, x_2) | -\infty < x_1 < \infty,\ 0 \le x_2 < \infty\}, \\ C &= \{(x_1, x_2) | -\infty < x_1 < \infty,\ x_2 = 0\}. \end{aligned}\right\} \tag{1.14}$$

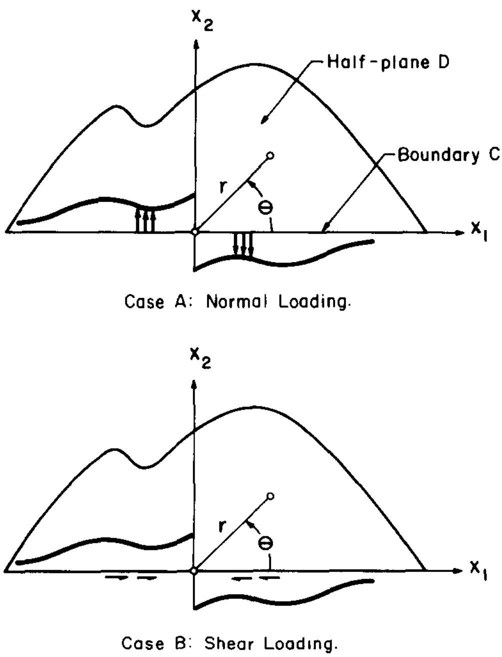

Fig. 1. Half-plane under discontinuous normal or shearing tractions.

* Note that the second of (1.10) implies $\tau_{\beta\alpha} = \tau_{\alpha\beta}$.

We suppose the boundary of the semi-infinite solid under consideration to be subjected to prescribed ordinary normal or shearing tractions, in the absence of applied couple-tractions. Consequently the boundary conditions (1.6) at present take on the two alternative forms given below.

Case A (*normal loading*)

$$\tau_{22}(x_1, 0) = p(x_1), \qquad \tau_{21}(x_1, 0) = 0, \qquad \sigma_2(x_1, 0) = 0 \qquad (-\infty < x_1 < \infty). \tag{1.15}$$

Case B (*shear loading*)

$$\tau_{21}(x_1, 0) = p(x_1), \qquad \tau_{22}(x_1, 0) = 0, \qquad \sigma_2(x_1, 0) = 0 \qquad (-\infty < x_1 < \infty). \tag{1.16}$$

The given load-function p, in either instance, is assumed to be sectionally smooth on any finite interval and absolutely integrable over the entire real axis; at each point of discontinuity the value of p is taken to be defined by the arithmetic mean of the corresponding right and left-hand limits. Finally, without essential loss in generality, we restrict p to be an odd function, i.e.

$$p(-x_1) = -p(x_1) \qquad (-\infty < x_1 < \infty). \tag{1.17}$$

Since D is now an unbounded domain, (1.15) or (1.16) need to be supplemented by regularity conditions at infinity, which we stipulate by requiring

$$\tau_{\alpha\beta}(x_1, x_2) = o(1), \qquad \sigma_\alpha(x_1, x_2) = o(1) \qquad \text{as } r \to \infty, \tag{1.18}$$

where $r = \sqrt{(x_\alpha x_\alpha)}$ is the distance from the origin.

The plane-strain problem for D governed by (1.15) or (1.16), together with (1.18), and the field equations (1.5) of the couple-stress theory, is a special case of the half-plane problem treated in Section 3 of [1]*. The solution to this problem was established there on the basis of the complete system of stress equations (1.7) with the aid of the generalized Airy stress functions introduced by Mindlin in [3] and by recourse to the exponential Fourier transform. We cite next from [1]† the results appropriate to the ordinary stress field, the couple-stress field, and the rotation field‡. In the interest of clarity we shall henceforth make explicit reference to the dependence of these fields upon the material length-parameter l and thus write $\tau_{\alpha\beta}(x_1, x_2; l)$ in place of $\tau_{\alpha\beta}(x_1, x_2)$, etc.

Case A (*normal loading*)

$$\left.\begin{aligned}
\tau_{\alpha\alpha}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s)a_{\alpha\alpha}(s, x_2; l)\sin(x_1 s)\,ds \qquad \text{(no sum)},\\
\tau_{\alpha\beta}(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s)a_{\alpha\beta}(s, x_2; l)\cos(x_1 s)\,ds \qquad (\alpha \neq \beta),\\
\sigma_1(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s)a_1(s, x_2; l)\sin(x_1 s)\,ds,
\end{aligned}\right\} \tag{1.19}$$

* In [1] the load-function was not restricted to be odd.

† Note that the present choice of coordinates corresponds to a rotation through $\pi/2$ of the frame chosen in [1].

‡ The rotation field was not exhibited in [1] but is immediately deducible from the results given in [1].

$$\left.\begin{aligned}\sigma_2(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s) a_2(s, x_2; l)\cos(x_1 s)\,ds,\\ \omega(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s) a(s, x_2; l)\cos(x_1 s)\,ds.\end{aligned}\right\} \quad \text{(1.19 contd.)}$$

Case B (shear loading)

$$\left.\begin{aligned}\tau_{\alpha\alpha}(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s) b_{\alpha\alpha}(s, x_2; l)\cos(x_1 s)\,ds, && \text{(no sum)},\\ \tau_{\alpha\beta}(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s) b_{\alpha\beta}(s, x_2; l)\sin(x_1 s)\,ds && (\alpha \neq \beta),\\ \sigma_1(x_1, x_2; l) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s) b_1(s, x_2; l)\cos(x_1 s)\,ds,\\ \sigma_2(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s) b_2(s, x_2; l)\sin(x_1 s)\,ds,\\ \omega(x_1, x_2; l) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s) b(s, x_2; l)\sin(x_1 s)\,ds.\end{aligned}\right\} \quad (1.20)$$

Here $\hat{p}$ designates the Fourier sine-transform of p, given by

$$\hat{p}(s) = \int_0^\infty p(x)\sin(sx)\,dx \qquad (0 \leq s < \infty). \quad (1.21)$$

The auxiliary functions $a_{\alpha\beta}$, a_α, a and $b_{\alpha\beta}$, b_α, b appearing in (1.19) and (1.20) are defined by

$$\left.\begin{aligned}a_{11}(s, x_2; l) = {} & \{(2\alpha - \beta - \alpha x_2 s)\exp(-x_2 s)\\ & -4(1-\nu)l^2 s^2 \alpha[\exp(-\alpha x_2/l) - \exp(-x_2 s)]\}\frac{1}{\beta},\\ a_{22}(s, x_2; l) = {} & \{(\beta + \alpha x_2 s)\exp(-x_2 s)\\ & +4(1-\nu)l^2 s^2 \alpha[\exp(-\alpha x_2/l) - \exp(-x_2 s)]\}\frac{1}{\beta},\\ a_{12}(s, x_2; l) = {} & \{(\beta - \alpha + \alpha x_2 s)\exp(-x_2 s)\\ & +4(1-\nu)ls\alpha[\alpha\exp(-\alpha x_2/l) - ls\exp(-x_2 s)]\}\frac{1}{\beta},\\ a_{21}(s, x_2; l) = {} & \{(\beta - \alpha + \alpha x_2 s)\exp(-x_2 s)\\ & +4(1-\nu)l^2 s^2[ls\exp(-\alpha x_2/l) - \alpha\exp(-x_2 s)]\}\frac{1}{\beta},\end{aligned}\right\} \quad (1.22)$$

$$\left.\begin{aligned}
a_1(s, x_2\,; l) &= 4(1-\nu)l^2 s[ls\exp(-\alpha x_2/l)-\alpha\exp(-x_2 s)]\frac{1}{\beta},\\
a_2(s, x_2\,; l) &= -4(1-\nu)l^2 s\alpha[\exp(-\alpha x_2/l)-\exp(-x_2 s)]\frac{1}{\beta},\\
a(s, x_2\,; l) &= \frac{1-\nu}{\mu}[ls\exp(-\alpha x_2/l)-\alpha\exp(-x_2 s)]\frac{1}{\beta},
\end{aligned}\right\} \quad \text{(1.22 contd.)}$$

$$\left.\begin{aligned}
b_{11}(s, x_2\,; l) &= \{(2-x_2 s)\exp(-x_2 s)\\
&\quad -4(1-\nu)l^2 s^2[\exp(-\alpha x_2/l)-\exp(-x_2 s)]\}\frac{\alpha}{\beta},\\
b_{22}(s, x_2\,; l) &= \{x_2 s\exp(-x_2 s)\\
&\quad +4(1-\nu)l^2 s^2[\exp(-\alpha x_2/l)-\exp(-x_2 s)]\}\frac{\alpha}{\beta},\\
b_{12}(s, x_2\,; l) &= \{(1-x_2 s)\exp(-x_2 s)\\
&\quad -4(1-\nu)ls[\alpha\exp(-\alpha x_2/l)-ls\exp(-x_2 s)]\}\frac{\alpha}{\beta},\\
b_{21}(s, x_2\,; l) &= \{\alpha(1-x_2 s)\exp(-x_2 s)\\
&\quad -4(1-\nu)l^2 s^2[ls\exp(-\alpha x_2/l)-\alpha\exp(-x_2 s)]\}\frac{1}{\beta},\\
b_1(s, x_2\,; l) &= 4(1-\nu)l^2 s[ls\exp(-\alpha x_2/l)-\alpha\exp(-x_2 s)]\frac{1}{\beta},\\
b_2(s, x_2\,; l) &= 4(1-\nu)l^2 s[\exp(-\alpha x_2/l)-\exp(-x_2 s)]\frac{\alpha}{\beta},\\
b(s, x_2\,; l) &= -\frac{1-\nu}{\mu}[ls\exp(-\alpha x_2/l)-\alpha\exp(-x_2 s)]\frac{1}{\beta},
\end{aligned}\right\} \quad (1.23)$$

in which

$$\alpha \equiv \alpha(ls) = (1+l^2s^2)^{\frac{1}{2}}, \qquad \beta \equiv \beta(ls) = \alpha(ls)+4(1-\nu)l^2s^2[\alpha(ls)-ls]. \quad (1.24)$$

Suppose temporarily the load-function p, in addition to meeting the previously imposed regularity conditions, is *continuous* on $(-\infty, \infty)$. Then it is not difficult to verify that the fields $\tau_{\alpha\beta}$, σ_α, ω given by (1.19), (1.20) for every $l > 0$, are continuous on $\bar{D}$, possess continuous partial derivatives of all orders on D, there satisfy (1.7) and the second of (1.8), and conform to the respective boundary conditions (1.15), (1.16) on C, as well as to the requirements (1.18) at infinity. Moreover, these properties uniquely characterize $\tau_{\alpha\beta}$, σ_α on $\bar{D}$ and render ω unique but for an arbitrary additive constant. The validity of the preceding solution for *discontinuous* loadings is readily confirmed by an appropriate limit process: it coincides with the limit of the sequence of solutions corresponding to a sequence of continuous loadings $\{p_n\}$ that tends to the given discontinuous p as $n \to \infty$. If p is discontinuous, the fields predicted by (1.19), (1.20) are once again found to satisfy the requisite field equations, boundary conditions, and regularity conditions at infinity. In this instance, however, $\tau_{\alpha\beta}$, σ_α, and ω at each point of load-discontinuity exhibit a singular behavior which will be examined in the next section.

We now proceed to the limit as $l \to 0$ in (1.19), (1.20) and to this end adopt the notation

$$\left.\begin{aligned} \mathring{\tau}_{\alpha\beta}(x_1, x_2) = \tau_{\alpha\beta}(x_1, x_2; 0+), \qquad \mathring{\sigma}_\alpha(x_1, x_2) = \sigma_\alpha(x_1, x_2; 0+), \\ \mathring{\omega}(x_1, x_2) = \omega(x_1, x_2; 0+). \end{aligned}\right\} \tag{1.25}$$

Taking this limit under the integral signs, as is permissible, we arrive at the subsequent results, which hold true for all (x_1, x_2) in D and at all points of C where p is continuous.

Case A (*normal loading*)

$$\left.\begin{aligned} \mathring{\tau}_{11}(x_1, x_2) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s)(1-x_2 s)\exp(-x_2 s)\sin(x_1 s)\,ds, \\ \mathring{\tau}_{22}(x_1, x_2) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s)(1+x_2 s)\exp(-x_2 s)\sin(x_1 s)\,ds, \\ \mathring{\tau}_{12}(x_1, x_2) = \mathring{\tau}_{21}(x_1, x_2) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s)x_2 s\exp(-x_2 s)\cos(x_1 s)\,ds, \\ \mathring{\sigma}_1(x_1, x_2) = \mathring{\sigma}_2(x_1, x_2) &= 0, \\ \mathring{\omega}(x_1, x_2) &= -\frac{2(1-\nu)}{\pi\mu}\int_0^\infty \hat{p}(s)\exp(-x_2 s)\cos(x_1 s)\,ds. \end{aligned}\right\} \tag{1.26}$$

Case B (*shear loading*)

$$\left.\begin{aligned} \mathring{\tau}_{11}(x_1, x_2) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s)(2-x_2 s)\exp(-x_2 s)\cos(x_1 s)\,ds, \\ \mathring{\tau}_{22}(x_1, x_2) &= -\frac{2}{\pi}\int_0^\infty \hat{p}(s)x_2 s\exp(-x_2 s)\cos(x_1 s)\,ds, \\ \mathring{\tau}_{12}(x_1, x_2) = \mathring{\tau}_{21}(x_1, x_2) &= \frac{2}{\pi}\int_0^\infty \hat{p}(s)(1-x_2 s)\exp(-x_2 s)\sin(x_1 s)\,ds, \\ \mathring{\sigma}_1(x_1, x_2) = \mathring{\sigma}_2(x_1, x_2) &= 0, \\ \mathring{\omega}(x_1, x_2) &= -\frac{2(1-\nu)}{\pi\mu}\int_0^\infty \hat{p}(s)\exp(-x_2 s)\sin(x_1 s)\,ds. \end{aligned}\right\} \tag{1.27}$$

The stress field $\mathring{\tau}_{\alpha\beta}$ satisfies the classical equilibrium and compatibility equations (1.12), while $\mathring{\tau}_{\alpha\beta}$ and $\mathring{\omega}$ obey the second of (1.13), as is easily verified. Further, $\mathring{\tau}_{\alpha\beta}$ meets the conventional stress boundary conditions in (1.15), (1.16) and is consistent with the regularity requirement at infinity expressed by the first of (1.18). Finally, the validity of the solution (1.26), (1.27) to the classical plane-strain problem governed by (1.15) or (1.16) together with the first (1.18) for *discontinuous* loadings may be justified by the limit process sketched earlier in connection with the analogous validation of the modified solution. Consequently, the modified solution represented by (1.19), (1.20) passes over into the corresponding classical solution* (1.26), (1.27), as $l \to 0$, at all points of $\bar{D}$ with the exception of the points of load-discontinuity on C, where neither solution is defined.

2. INVESTIGATION OF THE SINGULARITIES AT LOAD-DISCONTINUITIES

We turn now to our main objective and examine the singular behavior of both the modified and the classical solution given in Section 1 at points of the boundary where the prescribed normal or shearing tractions exhibit a finite jump discontinuity. With a view toward obtaining closed elementary representations for the dominant parts of the singularities to be explored we subject the load-function to certain additional regularity conditions, beyond those adopted in Section 1. These supplementary smoothness hypotheses may be summarized as follows: p is three times continuously differentiable on the open interval $(0, \infty)$; the first three derivatives of p are bounded and absolutely integrable on $(0, \infty)$. Consequently p is permitted to have at most a single jump discontinuity, the latter being situated at the origin (Fig. 1). Indeed, let

$$p(0+) = p_0 \neq 0. \tag{2.1}$$

Since the integrands in (1.19), (1.20) and (1.26). (1.27) are continuous functions of $(x_1, x_2; s)$ for all (x_1, x_2) in $\bar{D}$ and for $0 \leq s < \infty$, the desired singularities must stem from the behavior of these integrands as $s \to \infty$. We are therefore led to study the asymptotic character at large values of s of the functions entering the integral representation of the solutions under consideration. From (1.21) one infers through integration by parts that in the present circumstances

$$\hat{p}(s) = \frac{p_0}{s} + O(s^{-3}) \quad \text{as } s \to \infty. \tag{2.2}\dagger$$

Next, (1.24) imply

$$\left.\begin{aligned} \alpha(s) &= s + \frac{1}{2s} - \frac{1}{8s^3} + O(s^{-5}), \\ \beta(s) &= (3-2\nu)s + \frac{\nu}{s} + O(s^{-3}) \quad \text{as } s \to \infty. \end{aligned}\right\} \tag{2.3}$$

Further, (1.22) and (1.23), in view of (2.3), furnish the asymptotic expansions listed below, all of which are valid for fixed $l > 0$ and every $x_2 \geq 0$, in the limit as $s \to \infty$:

$$\left.\begin{aligned} a_{11}(s, x_2; l) &= \frac{\exp(-x_2 s)}{3-2\nu}\left[(1-2\nu)x_2 s - (1-2\nu) - \frac{(1-\nu)x_2^2}{2l^2} + O(s^{-1})\right], \\ a_{22}(s, x_2; l) &= \frac{\exp(-x_2 s)}{3-2\nu}\left[-(1-2\nu)x_2 s + (3-2\nu) + \frac{(1-\nu)x_2^2}{2l^2} + O(s^{-1})\right], \\ a_{12}(s, x_2; l) &= \frac{\exp(-x_2 s)}{3-2\nu}\left[-(1-2\nu)x_2 s + 4(1-\nu) + \frac{(1-\nu)x_2^2}{2l^2} + O(s^{-1})\right], \\ a_{21}(s, x_2; l) &= \frac{\exp(-x_2 s)}{3-2\nu}\left[-(1-2\nu)x_2 s + \frac{(1-\nu)x_2^2}{2l^2} + O(s^{-1})\right], \end{aligned}\right\} \tag{2.4}$$

* See also Sneddon [5], Art. 45.2, where the conventional solution for Case A is deduced directly.
† See also Erdélyi [6], p. 47.

$$\left.\begin{aligned}
a_1(s,x_2;l) &= \frac{2(1-\nu)l\exp(-x_2s)}{3-2\nu}\left[-\frac{x_2}{l}-\left(1-\frac{x_2^2}{4l^2}\right)\frac{1}{ls}+O(s^{-2})\right],\\
a_2(s,x_2;l) &= \frac{2(1-\nu)l\exp(-x_2s)}{3-2\nu}\left[\frac{x_2}{l}-\frac{x_2^2}{4l^3s}+O(s^{-2})\right],\\
a(s,x_2;l) &= \frac{(1-\nu)\exp(-x_2s)}{2(3-2\nu)\mu}\left[-\frac{x_2}{l^2s}-\left(1-\frac{x_2^2}{4l^2}\right)\frac{1}{l^2s^2}+O(s^{-3})\right];
\end{aligned}\right\} \quad (2.4\ \text{contd.})$$

$$\left.\begin{aligned}
b_{11}(s,x_2;l) &= \frac{\exp(-x_2s)}{3-2\nu}\left[(1-2\nu)x_2s+2-\frac{(1-\nu)x_2^2}{2l^2}+O(s^{-1})\right],\\
b_{22}(s,x_2;l) &= \frac{\exp(-x_2s)}{3-2\nu}\left[-(1-2\nu)x_2s+\frac{(1-\nu)x_2^2}{2l^2}+O(s^{-1})\right],\\
b_{12}(s,x_2;l) &= \frac{\exp(-x_2s)}{3-2\nu}\left[(1-2\nu)x_2s-(1-2\nu)-\frac{(1-\nu)x_2^2}{2l^2}+O(s^{-1})\right],\\
b_{21}(s,x_2;l) &= \frac{\exp(-x_2s)}{3-2\nu}\left[(1-2\nu)x_2s+(3-2\nu)-\frac{(1-\nu)x_2^2}{2l^2}+O(s^{-1})\right],\\
b_1(s,x_2;l) &= \frac{2(1-\nu)l\exp(-x_2s)}{3-2\nu}\left[-\frac{x_2}{l}-\left(1-\frac{x_2^2}{4l^2}\right)\frac{1}{ls}+O(s^{-2})\right],\\
b_2(s,x_2;l) &= \frac{2(1-\nu)l\exp(-x_2s)}{3-2\nu}\left[-\frac{x_2}{l}+\frac{x_2^2}{4l^3s}+O(s^{-2})\right],\\
b(s,x_2;l) &= \frac{(1-\nu)\exp(-x_2s)}{2(3-2\nu)\mu}\left[\frac{x_2}{l^2s}+\left(1-\frac{x_2^2}{4l^2}\right)\frac{1}{l^2s^2}+O(s^{-3})\right].
\end{aligned}\right\} \quad (2.5)$$

As a final preliminary to our present task we recall* the familiar integral representations

$$\left.\begin{aligned}
&\int_0^\infty \exp(-x_2s)\cos(x_1s)\,ds = \frac{x_2}{r^2}, \quad \int_0^\infty \exp(-x_2s)\sin(x_1s)\,ds = \frac{x_1}{r^2},\\
&\int_0^\infty s\exp(-x_2s)\cos(x_1s)\,ds = \frac{x_2^2-x_1^2}{r^2}, \quad \int_0^\infty s^{-1}\exp(-x_2s)\sin(x_1s)\,ds = \frac{\pi}{2}-\theta,
\end{aligned}\right\} \quad (2.6)$$

as well as the estimate

$$\int_1^\infty s^{-1}\exp(-x_2s)\cos(x_1s)\,ds = -\log r+O(1) \text{ as } r\to 0, \quad (2.7)$$

in which (r,θ) are the polar coordinates defined by

$$r = (x_1^2+x_2^2)^{\frac{1}{2}}, \qquad \theta = \tan^{-1}(x_2/x_1) \qquad (0\le\theta\le\pi). \quad (2.8)$$

Equations (2.6), (2.7) hold true for all (x_1, x_2) on D.

* See [1], Section 4.

From (1.19), (1.20), in conjunction with (2.2), (2.4), (2.5), (2.6), (2.7) one draws the subsequent estimates appropriate to the *modified solution* for every fixed $l > 0$, in the limit as $r \to 0$.

Case A (*normal loading*)

$$\left.\begin{aligned}
&\tau_{11}(x_1, x_2; l) = \frac{(1-2\nu)p_0}{(3-2\nu)\pi}(-\pi+2\theta+\sin 2\theta)+o(1),\\
&\tau_{22}(x_1, x_2; l) = \frac{p_o}{\pi}\left(\pi-2\theta-\frac{1-2\nu}{3-2\nu}\sin 2\theta\right)+o(1),\\
&\tau_{12}(x_1, x_2; l) = \frac{8(1-\nu)p_0}{(3-2\nu)\pi}\log r+O(1), \qquad \tau_{21}(x_1, x_2; l) = \frac{2(1-2\nu)p_0}{(3-2\nu)\pi}\sin^2\theta+o(1),\\
&\sigma_1(x_1, x_2; l) = o(1), \qquad \sigma_2(x_1, x_2; l) = o(1), \qquad \omega(x_1, x_2; l) = \omega(0, 0; l)+o(1),
\end{aligned}\right\} \tag{2.9}$$

where

$$\omega(0, 0; l) = \frac{2(1-\nu)}{\pi\mu}\int_0^\infty \frac{ls-\alpha(ls)}{\beta(ls)}\hat{p}(s)\,ds. \tag{2.10*}$$

Case B (*shear loading*)

$$\left.\begin{aligned}
&\tau_{11}(x_1, x_2; l) = \frac{4p_0}{(3-2\nu)\pi}\log r+O(1), \qquad \tau_{22}(x_1, x_2; l) = \frac{2(1-2\nu)p_0}{(3-2\nu)\pi}\sin^2\theta+o(1),\\
&\tau_{12}(x_1, x_2; l) = \frac{(1-2\nu)p_o}{(3-2\nu)\pi}(-\pi+2\theta+\sin 2\theta)+o(1),\\
&\tau_{21}(x_1, x_2; l) = \frac{p_0}{\pi}\left(\pi-2\theta+\frac{1-2\nu}{3-2\nu}\sin 2\theta\right)+o(1),\\
&\sigma_1(x_1, x_2; l) = \sigma_1(0, 0; l)+o(1), \qquad \sigma_2(x_1, x_2; l) = o(1), \qquad \omega(x_1, x_2; l) = o(1),
\end{aligned}\right\} \tag{2.11}$$

where

$$\sigma_1(0, 0; l) = \frac{8(1-\nu)l^2}{\pi}\int_0^\infty \frac{ls-\alpha(ls)}{\beta(ls)}s\hat{p}(s)\,ds. \tag{2.12}$$

Similarly, (1.26), (1.27), together with (2.2), (2.6), (2.7), yield the analogous estimates for the *classical solution* in the limit as $r \to 0$.

Case A (*normal loading*)

$$\left.\begin{aligned}
&\mathring{\tau}_{11}(x_1, x_2) = \frac{p_0}{\pi}(\pi-2\theta-\sin 2\theta)+o(1), \qquad \mathring{\tau}_{22}(x_1, x_2) = \frac{p_0}{\pi}(\pi-2\theta+\sin 2\theta)+o(1),\\
&\mathring{\tau}_{12}(x_1, x_2) = \mathring{\tau}_{21}(x_1, x_2) = -\frac{2p_0}{\pi}\sin^2\theta+o(1), \qquad \mathring{\omega}(x_1, x_2) = \frac{2(1-\nu)p_0}{\pi\mu}\log r+O(1).
\end{aligned}\right\} \tag{2.13}$$

* Observe from (2.2), (2.3) that this improper integral is convergent for every $l > 0$. Note that the term $\omega(0, 0; l)$ in the last of (2.9) may be omitted since it represents an inessential rigid rotation of the entire body.

Case B (shear loading)

$$\left.\begin{aligned} &\mathring{\tau}_{11}(x_1,x_2) = \frac{4p_0}{\pi}\log r + O(1), \qquad \mathring{\tau}_{22}(x_1,x_2) = -\frac{2p_0}{\pi}\sin^2\theta + o(1), \\ &\mathring{\tau}_{12}(x_1,x_2) = \mathring{\tau}_{21}(x_1,x_2) = \frac{p_0}{\pi}(\pi - 2\theta - \sin 2\theta) + o(1), \\ &\mathring{\omega}(x_1,x_2) = -\frac{(1-\nu)p_0}{\pi\mu}(\pi - 2\theta) + o(1). \end{aligned}\right\} \tag{2.14}$$

Equations (2.11) are consistent with, but more refined than, the analogous estimates deduced in Section 5 of [1] for a uniformly distributed shear loading that is confined to a finite segment of the boundary.* We note from (2.9), (2.11) that both the couple-stress field and the rotation field in either loading case remain finite and continuous on the closed half-plane $\bar{D}$ according to the modified theory. In contrast, as is evident from (2.13), (2.14), the rotation field predicted by the classical theory has a logarithmic singularity at the origin in Case A and is discontinuous there in Case B. This conclusion regarding the behavior of ω at the load-discontinuity confirms a mitigating influence of couple-stresses in the singular problem under consideration, which is characteristic of all of the singular problems studied in [1] and [2].

Turning to the discussion of the ordinary stress fields $\tau_{\alpha\beta}$ and $\mathring{\tau}_{\alpha\beta}$, we observe first that whereas $\mathring{\tau}_{\alpha\beta}$ is of course independent of the elastic constants, $\tau_{\alpha\beta}$ involves Poisson's ratio ν in addition to the material length-parameter l. In Case *B* both τ_{11} and $\mathring{\tau}_{11}$ display a singularity of the same (logarithmic) order at the origin. The remaining stress components exhibit merely a discontinuity at $r = 0$: their limit as $r \to 0$ depends on the inclination of the ray $\theta =$ constant along which the origin is approached. On the other hand, in Case A the shear stress τ_{12} is seen to become logarithmically unbounded as $r \to 0$, despite the fact that its classical counterpart $\mathring{\tau}_{12}$ is merely discontinuous at the origin; the remaining stress components are finite—though discontinuous—at the origin according to both theories.

The foregoing conclusion concerning the behavior of τ_{12} in Case A represents a striking departure from the results encountered in [1], [2] and furnishes the primary motivation of the present paper. It dispells the intuitively plausible notion supported by [1], [2] that ordinary stresses which are finite in the classical solution of a singular problem necessarily remain bounded when couple-stresses are taken into account. As is now apparent, a sufficiently sharp variation in applied ordinary normal tractions may, in the presence of couple-stresses, give rise to an arbitrarily large local *amplification* of a conventional shear stress—regardless of the relative size of the characteristic length-parameter. This fact would seem to be of considerable interest in connection with efforts to appraise the physical significance of the couple-stress theory.

It is essential to emphasize that formulas (2.9), (2.11) and (2.13), (2.14), which predict qualitative differences in the singular behavior at the load-discontinuity of the modified solution for every *fixed* $l > 0$ as compared to the classical solution, in no way contradict our assertion at the end of Section 1 concerning the transition from the modified to the classical solution as $l \to 0$. For the purpose of clarifying this issue further we call $\bar{D}'$ the complement of the origin with respect to $\bar{D}$, i.e. set

$$\bar{D}' = \{(x_1,x_2) \mid -\infty < x_1 < \infty,\ 0 \le x_2 < \infty,\ (x_1,x_2) \ne (0,0)\}, \tag{2.15}$$

* In [1] contributions to $\tau_{\alpha\beta}$ and σ_α that remain bounded at the endpoints of the load-interval were not determined explicitly, while the behavior of ω at the points of load-discontinuity was not examined.

and focus our attention on a particular component of ordinary stress—say τ_{11} and $\mathring{\tau}_{11}$. In either loading case

$$\lim_{l\to 0}\,[\tau_{11}(x_1, x_2; l)/\mathring{\tau}_{11}(x_1, x_2)] = 1 \text{ on } \bar{D}', \tag{2.16}$$

the convergence being non-uniform with respect to (x_1, x_2). From (2.16) follows

$$\lim_{r\to 0}\lim_{l\to 0}\,[\tau_{11}(x_1, x_2; l)/\mathring{\tau}_{11}(x_1, x_2)] = 1. \tag{2.17}$$

On the other hand, as is clear from (2.9), (2.13), for Case A,

$$\lim_{l\to 0}\lim_{r\to 0}\,[\tau_{11}(x_1, x_2; l)/\mathring{\tau}_{11}(x_1, x_2)] = -\frac{1-2\nu}{3-2\nu}, \tag{2.18}$$

while from (2.11), (2.14), in Case B,

$$\lim_{l\to 0}\lim_{r\to 0}\,[\tau_{11}(x_1, x_2; l)/\mathring{\tau}_{11}(x_1, x_2)] = \frac{1}{3-2\nu}. \tag{2.19}$$

Equations (2.17), (2.18), (2.19) reveal the discontinuous dependence upon l at $l = 0$ of the stress-ratio $\tau_{11}/\mathring{\tau}_{11}$ in the limit as $r \to 0$. This observation accounts for appreciable departures from the classical values of the normal stress under consideration at arbitrarily small values of l in a sufficiently small neighborhood of the singular point. Analogous statements apply to the remaining stress components, as well as to the rotation. The situation just described is typical of the severe boundary-layer effects encountered in all of the singular stress-concentration problems treated in [1] and [2].

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Résumé—Les singularités induites par des tractions discontinues normales ou de *cisaillement* appliquées à un solide semi-infini sont examinées dans le contexte de la théorie du comportement élastique couple-tension linéarisé à deux dimensions et sont comparées à leur pendant dans la théorie classique de la déformation plane. Un saut limité dans la charge de *cisaillement* est trouvé pour produire une infinité logarithmique à la discontinuité de charge dans la tension *normale* agissant parallèlement à la délimitation selon les deux théories. Faisant contraste, un saut limité dans la charge *normale*, selon la théorie couple-tension, donne lieu à une infinité logarithmique à la discontinuité de charge dans la tension de *cisaillement* perpendiculairement à la délimitation, quoiques toutes les tensions restent limitées dans la solution classique correspondante; tandis que la théorie conventionnelle pour ce cas de charge prédit une singularité logarithmique dans le champ de rotation, ce dernier restant limité dans la solution modifiée.

Zusammenfassung—Die Singularitäten die von unstetigen Normalund Schubbelastungen herrühren werden im Rahmen einer linearen Elastizitätstheorie die Momentspannungen berücksichtigt untersucht und mit den entsprechenden klassischen Singularitäten verglichen. Es zeigt sich dass ein endlicher Sprung in der Normalbelastung am Rande einer Halbebene eine logarithmische Singularität in jener Schubspannung hervorruft die senkrecht zum•Rande wirkt, obwohl das klassische Spannungsfeld für eine solche Belastung völlig beschränkt bleibt. Im Gegensatz bleibt das Drehungsfeld, das in der klassischen Lösung logarithmisch unendlich wird an der Unstetigkeitsstelle, beschränkt in der erweiterten Theorie.

Абстракт—Исследуется сингулярности, вызываемые разрывными моментными напряжениями, нормальными или сдвига, приложенными к полубесконечному твердому телу, при использовании двухмерной, линеаризованной теории моментных напряжений упругого поведения. Сравнивается с их эквивалентами в классической теории плоской деформации. Найдено, что конечный скачок при нагрузке *сдвигом* вызывает логаритмическую бесконечность при разрыве нагрузки в *нормальном* напряжении, действующим параллельно контуру, в согласии с двумя теориями. В противоположность, конечный скачок при *нормальной* нагрузке, в согласии с теорией моментных напряжений, вызывает рост логаритмической бесконечности при разрыве нагрузки в напряжении *сдвига*, при простых углах к контуру, несмотря на это, все напряжения ограничени соответствующим классическим решением. Но там, где обыкновенная теория для этого случая нагрузки предусматривает логаритмическую сингулярность во врашающемся поле, позже остаются ограниченной в модифицированном решении.